Probabilistic Modelling: A Few Case Studies in Data Analysis and Performance Analysis

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Acknowledgement

- Dr. Yuwei Xu
- Dr. Suet-Peng Yong
- Other collaborators
- Other inspiring researchers

Outline

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Background

- Probability concepts have been around for some time.
  - **Democritus:** *Everything existing in the universe is the fruit of chance.*
  - **Boethius:** *Chance, too, which seems to rush along with slack reins, is bridled and governed by law.*
  - **Caesar,** Julius: *Iacta alea est. (The die is cast.)*
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- 宋人方岳：“不如意事常八九，可与人語無二三。”
- Pioneers: Leibniz, Pascal
- **Laplace,** Pierre Simon: *It is remarkable that a science which began with the consideration of games of chance should have become the most important object of human knowledge.*

Probability in Information Sciences

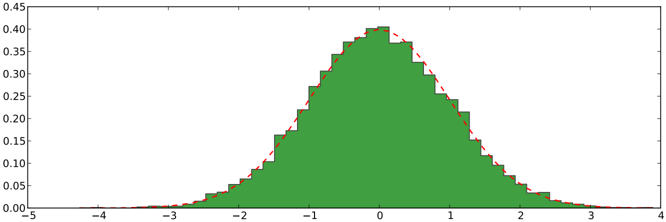
- Artificial Intelligence
  - Probabilistic inference
  - Decisions under partial information
  - Processing signals (e.g., speech, images)
- Computer Networks
  - Channel scheduling
  - Packet collision
  - Queueing behaviour at routers
- Software Engineering
  - Model failure of safety-critical systems
- Data Compression
  - Shannon Theorem
  - Huffman coding

Outline

Common Probability Distributions

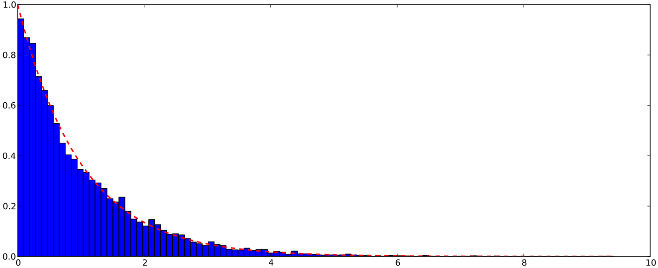
- There are continuous and discrete probability distributions.
  - Uniform distribution
  - Normal distribution (*aka* Gaussian)
  - Bernoulli distribution
  - Binomial distribution
  - Poisson distribution
  - Exponential distribution
  - Pareto distribution

Gaussian Distribution



- Most prominent distribution in statistics.
- Central limit theorem: under mild conditions the sum of a large number of random variables is distributed approximately normally.
- pdf: $p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

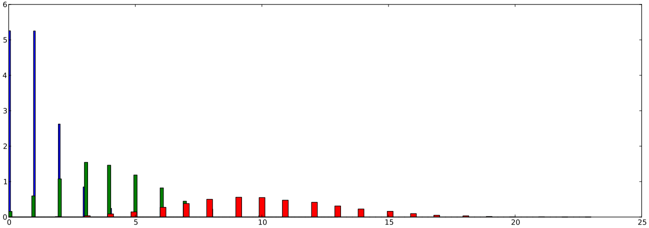
Exponential Distribution



- $p(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$ *and* $F(x) = 1 - e^{-\lambda x}$
- $E(X) = 1/\lambda,\ Var(X) = 1/\lambda^2$
- Models
  - Life span of equipments, call duration, job processing time ...

$\Rightarrow$ Question: How likely does it last longer than average?

Poisson Distribution



- Models random occurrence of *discrete* events.
  - Service requests received per hour.
  - Number of packets arriving at a node per second.
- $P(n) = \dfrac{e^{-\lambda}\lambda^n}{n!}, n = 0, 1, 2, \ldots$
- $E(n) = \lambda = Var(n)$

Age Distribution of Populations

- Guess how would it look like?

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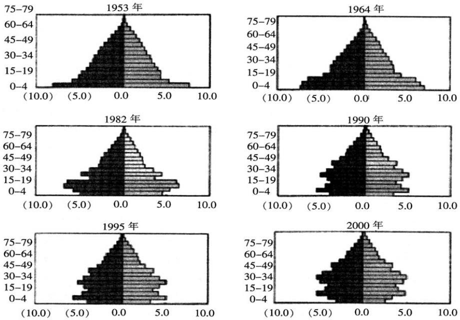

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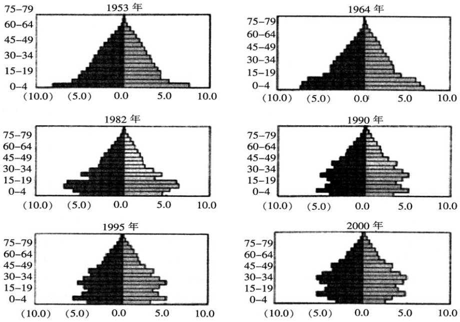


- Discussion point: Will the age distribution of the populations affect the outcome of population-based optimization?

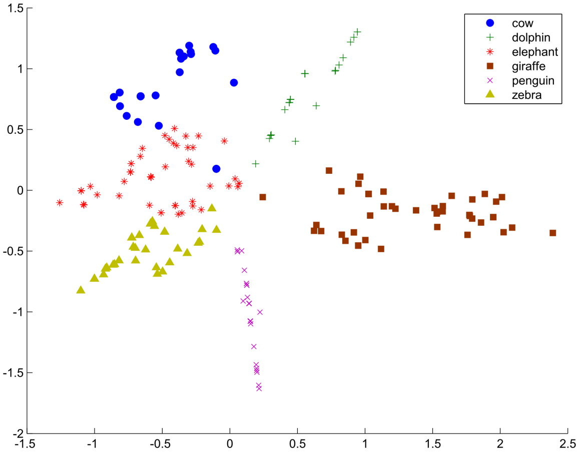
Outline

Clustering

- Clustering algorithms assume data distributions within the clusters.
- Gaussian E-M: assume Gaussian ellipsoids.
  - k-means is a special case of GEM.
- Mixture of Gaussian models can be explored for classification, and anomaly detection.

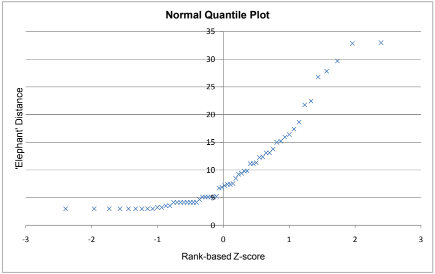
Case 1: Scene analysis and novelty detection

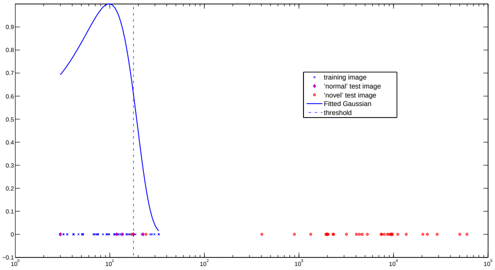
- S-P Yong et al., *Pattern Recog.*, 45(9), 2012.
- 14×14 label co-occurrence matrices as 'feature code'
- Clusters reviewed by PCA
- Are they Gaussian?



1-D Gaussian modeling

- High-dimensional space tricky to model
- Resort to modelling point-to-centre distances ☺
- Q-Q plot seems okay
- χ^2 goodness-of-fit test: p-value=0.085 (null hypothesis *not* rejected)
- A simple threshold used for outlier detection





The Quincunx

http://www.mathsisfun.com/data/quincunx.html

Case 2: Learning the k in k-means

- Lacking prior knowledge, often we don't know k.
- The idea: Start with a low number (e.g., 1), and examine the clusters.
- If a cluster passes Gaussian test, keep it; otherwise split.
- Questions:
  - *How* to test the normality?
  - *How* to split?

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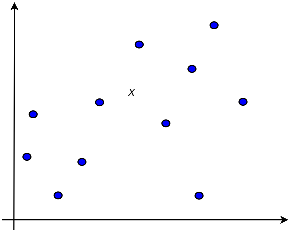
G-means (G. Hamerly & C. Elkan, NIPS 2003)

Algorithm 1 G-means(X, α)

Input: X - dataset, α - a confidence level
Output: k cluster centres C

1: Initialize C as a set of centres (usually $C \leftarrow \{\overline{x}\}$)
2: $C \leftarrow$ kmeans(C, X).
3: **for all** $c_j \in C$ **do**
4: Let $C^{(j)} = \{x_i | \text{class}(x_i) = j\}$ be the set of datapoints assigned to center c_j .
5: Use a statistical test to detect if each $C^{(j)}$ follows a Gaussian distribution (at confidence level α).
6: If the data look Gaussian, keep c_j. Otherwise replace c_j with two centres.
7: **end for**
8: **Repeat** from step 2 until no more centres are added.

How to test a cluster?



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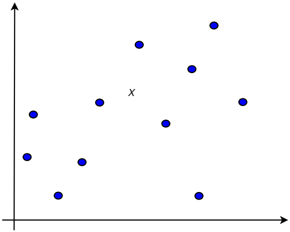

- Initialize two centres c_1 and c_2; re-cluster.
- Get $v = c_1 - c_2$, and project data vector x_i onto v: $x_i' = \langle v, x_i \rangle / \|v\|$. Normalize X' to zero mean and variance 1.
- Calculate empirical cumulative density function $z_i = F(x'_{(i)})$, and the Anderson-Darling statistics $A^2(Z) = -\frac{1}{n}\sum_i (2i-1)[\log(z_i) + \log(1 - z_{n+1-i})] - n$
- If statistics within range of the critical value, keep the original centre, and discard $\{c_1, c_2\}$; otherwise, replace the current centre with $\{c_1, c_2\}$.

Another Take on Initialization

- What if we want a one-shot clustering with k clusters?
- Requires a better way to do initialization.
- Authur & Vassilvitskii (2007): k-means++: The Advantages of Careful Seeding

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Algorithm 2 k-means++(X, k)

Input: X - dataset, k: number of cluster centres
Output: $C = \{c_i\}$: a set of k initial centres
1: Take one centre c_1, chosen uniformly at random from X.
2: Take a new centre c_i, choosing $x \in X$ with probability $\frac{D(x)^2}{\sum_{x \in X} D(x)^2}$.
3: Repeat Step 2 until we have taken k centres altogether

$D(x)$: the shortest distance from data point x to the closest centre.

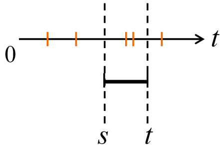
Outline

Stochastic Processes

- Over a continuous time parameter, SP is defined as a collection of random variables.
  - Denoted as $\{X_t\}$, $t \in R$.
- Over a discrete time parameter, is defined as a collection of random variables.
  - Denoted as $\{X_n\}$, $b \in Z$.
- These random variables are related and defined in the same probability space.
- Stationary stochastic process: statistics of the process will not vary over time.

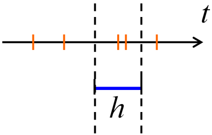
Point Process

- Point process (*aka* counting process), is a process with random occurrence of points on a line.
- Denoted as $\{N(t), t \geq 0\}$.
  - Number of customers arriving in a shop during time of $[0, t)$.
- If $s < t$, then $N(s) \leq N(t)$.
- Increment $N(s) - N(t)$: Number of event occurrence within (s, t).



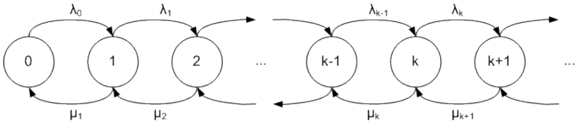
Poisson Process

- N has stationary increments.
- N has independent increments.
- Probability of 1 arrival in small interval h:
  - $P[N(h) = 1] = \lambda h + o(h)$.
- Probability of 2 or more arrivals in h:
  - $P[N(h) \geq 2] = o(h)$.
- Such a point process is a Poisson Process with a rate of $\mu > 0$.



Graphical Representation

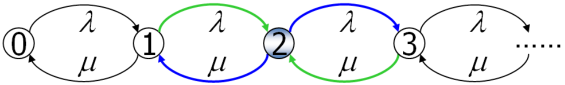
- Markov Chains are used to describe system state transition in a Poisson process.
- A point process counting arrivals only: always growing
- Birth-death process can be used for queueing modeling.
  - Right links represent birth or arrival;
  - Left links are for death or departure.



Kendall Notation ($A/B/c/K/m/Z$)

- A: interarrival time distribution
- B: service time distribution
  - M for exponential
  - G for general
- c: number of servers
- Optional:
  - K: maximum number of allowed customers
  - m: size of the customer population
  - Z: queueing discipline, typically FIFO

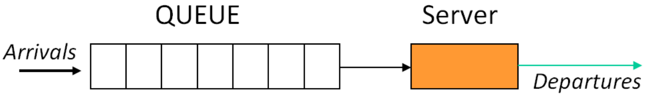
M/M/1 Queuing



- Interarrival time: exponentially distributed, mean$=1/\lambda$
- Job processing time: exponentially distributed, mean$=1/\mu$
- Steady state requires $\lambda < \mu$
- FIFO
- One server
- Chances the server is busy: $P[N \geq 1] = 1 - p_0 = \rho$.
- Expected number in system: $L = E\{N\} = \sum_n np_n = \frac{\rho}{1-\rho}$

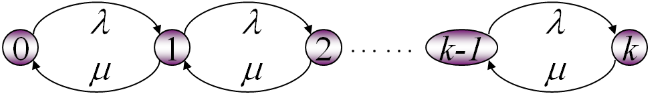
A Little Variation to $M/M/1$

- In M/M/1, there is no limit on the queue length.
- In reality, services usually don't support unlimited queueing (memory, ports etc.)
- If a customer finds no available position in a limited queue, it is supposed to disappear!



$M/M/1/K$ Analysis

- Transition diagram



- Steady state solutions:
  - $\sum_{n=0}^{K} p_n = 1$
  - $p_n = (\frac{\lambda}{\mu})^n p_0, 0 \leq n \leq K$ (K was ∞ in $M/M/1$).

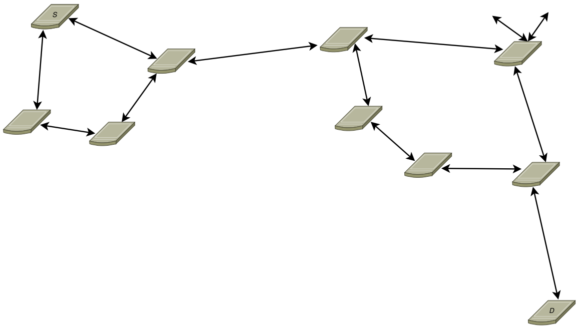
Expected Customer Numbers

- Solution to the probabilities:

$$p_n = \rho^n (1-\rho)/(1-\rho^{K+1}).$$

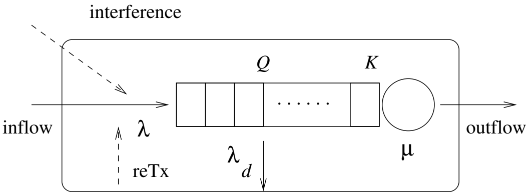
- Expected customer number in system:
  - $L = E\{N\} = \sum_{n=0}^{K} n p_n = \frac{\rho}{1-\rho} - \frac{(K+1)\rho^{K+1}}{1-\rho^{K+1}}$
- Probability that an arriving customer is rejected is (simply) p_K.
- Rejection rate is therefore $p_K \lambda$.
- **Actual** arrival rate into the system is
  - $\lambda' = (1-p_K)\lambda$.

Case 3: Video streaming on wireless multi-hop networks



- Each node subject to MAC contention, interference, load, limited buffer, and other overhead ...
- Treated as M/M/1/K to form a Jackson network
- What is the best routing metric: #hops, load, delay ...?

Our Problem(s)



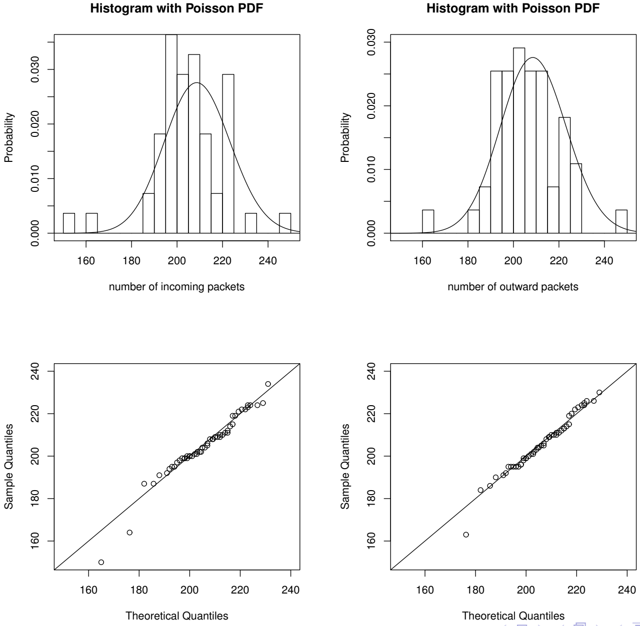
- In reality, we know λ, but not μ ...
- We do know Q ($Q \leq K$), and λ_d
- The question: how to estimate μ, so that the processing time for each node can be derived.
- Hang on - how do you know you are handling Poisson traffic?

Simulation Settings

- Network Simulator 2 (NS-2) + EvalVid in an IEEE 802.11b/g networks of grid topologies: 5×5, 7×7, 9×9, 15×15

Parameters	**Values**
Distance between two neighbours	150m
Antenna Type	Omnidirectional
Standard	802.11b
Transmission Range	250m
Transmission Rate	11 Mbits/s
Packet Size	1024 bytes
Queue Size	50 packets per node
Video Format	H.264/MEPG4
Duration	$29 \sim 66$s
Number of Streams	$3 \sim 7$
Minimum number of hops	4

Poisson Inflow / Outflow



Chi-Square Test Results

For video 'Highway':

Node	Inflow		Outflow	
	DF	χ^2	DF	χ^2
N30	6	12.1596	6	12.1198
N36	5	7.3989	6	10.6659
N31	6	11.1698	6	10.9301

For video 'Grandma':

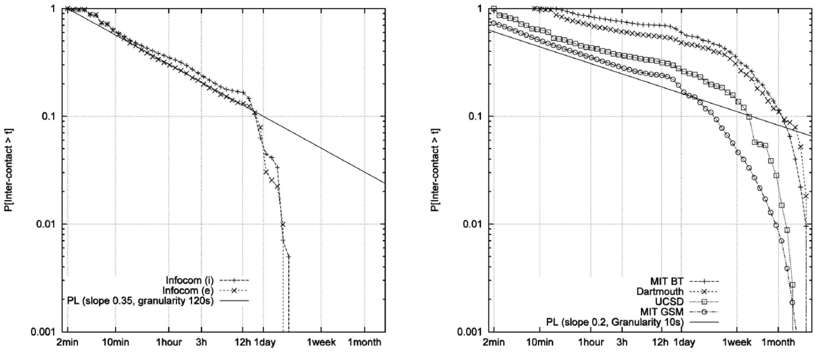
Node	Inflow		Outflow	
	DF	χ^2	DF	χ^2
N34	5	5.2467	5	1.4930
N41	5	4.3583	5	6.0237
N49	5	1.2288	5	4.3098

Our assumption on Poisson traffic holds on the significance level of $\alpha = 0.05$.

Case 4: Mobility Modelling and Simulation in DTNs

- Delay Tolerant Networks: opportunistic forwarding, e.g.,Vehicular ad hoc networks
- Simulators use simple motion models such as Random Walk and Random Waypoint etc.
- Real traces are available but few
- Question: how good are the motion models in simulations?
- Karagiannis et al. (MobiCom'07): Power law and exponential decay of inter contact times between mobile devices.

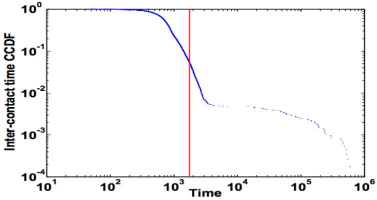
Power Law in Real Trace Data

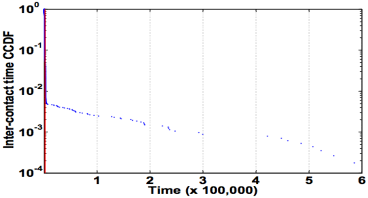


Dichotomy exists

- Good linear fitting up to day-level (power law)
- Then exponential delay follows

Dichotomy Generated by Simulated Motion Patterns





- ☺ Simple motion patterns (e.g. Random Waypoint) generate similar effects

Elsewhere

- Dirichlet distributions widely used for topic modelling in text and multimedia content analysis
- Markov chain models for weather and renewable energy data modelling and simulation
- ... (almost) the entire Machine Learning field
- ... Your contributions ☺

La vita è bella

美麗人生 - 1997